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SECONDARY COHOMOLOGY OPERATIONS INDUCED BY THE DIAGONAL MAPPING†

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INTRODUCTION

FOR SOME TIME it has been clear that higher order cohomology operations can be useful in studying homotopy problems. An outstanding instance is Adams' solution of Hopf's problem concerning the existence of elements of Hopf invariant one, by using stable secondary cohomology operations [1].

Until now, secondary cohomology operations studied have been almost exclusively operations of one variable. (The Massey triple and n-ary products are exceptions.) This paper introduces and studies *diagonal operations*, a new class of secondary cohomology operations of several variables. If a is a member of the mod p Steenrod algebra $\mathscr{A}_p$, then the diagonal operation $a_d(u_1 \times \dots \times u_n)$ is obtained by applying the functional cohomology operation a_d, induced by the diagonal mapping $d:(X,*) \longrightarrow (X^n, T^nX)$, to a cross product of elements $u_i \in H^*(X,*;Z_p)$, $i = 1, \dots, n$.

A secondary cohomology operation is usually associated with a relation on primary operations. In this sense, a_d is connected with the *Cartan formula* ([5] and [16] 6.12) of a. This connection can be seen in the domain and indeterminacy of $a_d(u_1 \times \dots \times u_n)$. More significantly, Theorems (2.3) and (2.4), which are analogues of Theorems (6.1) and (6.3) of Peterson and Stein [12], show that the diagonal operation $a_d(u_1 \times \dots \times u_n)$ can sometimes be expressed in terms of functional cohomology operations derived from the primary operations appearing in the Cartan formula of a. These theorems are applied in §2 to prove in certain cases that the composition gf of two essential mappings f and g is essential, by showing that a *functionalized diagonal operation* $(a_d)_{gf}$ (see Definition 2.9) is non-vanishing.

Whenever there is a non-vanishing diagonal operation of n variables in the cohomology of a space X, then the Liusternik–Shnirel'man category‡ of X is at least n. In §4, we show that diagonal operations are also related to Hilton's generalized Hopf invariants.

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‡ Normalized so that a contractible space has category zero.

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Theorem (3.1) expresses a diagonal operation of three variables in terms of diagonal operations of two variables associated with the Cartan formula. This is applied to sphere bundles over spheres. For example, cat $Sp(2) = 3$.

§1. DEFINITION OF DIAGONAL OPERATIONS

All spaces treated are in the category of spaces with base points (denoted by $*$) and mappings and homotopies which preserve the base point. In addition we require that every pair of spaces have the same homotopy type as a pair (X, Y), where X and Y are countable connected CW-complexes and Y is a subcomplex of X.

If (X, A) and (Y, B) are pairs of spaces, then the product $(X, A) \times (Y, B)$ is the pair $(X \times Y, X \times B \cup A \times Y)$. The nth power $(X,*)^n$ of a space $(X,*)$ is then defined recursively; it is easy to see that

$$(X, *)^n = (X^n, T^nX),$$

where T^nX consists of those points of X^n which have at least one coordinate equal to $*$. The *diagonal mapping*

$$d_n : (X, *) \longrightarrow (X, *)^n$$

is defined $d_n(x) = (x, \ldots, x)$. For simplicity, we usually write d for d_n.

Throughout the paper, singular cohomology with coefficients in the field $\mathbf{Z}_p$ of integers modulo the prime p will be used, even when the notation $H^*(X, A)$ does not indicate the coefficients. Then we have the cross product $u \times v = \alpha(u \otimes v)$,

$$\alpha : H^q(X, A) \otimes H^r(Y, B) \longrightarrow H^{q+r}(X \times Y, X \times B \cup A \times Y),$$

and the *cup product* $u \cup v = d^*\alpha(u \otimes v) = d^*(u \times v)$,

$$d^*\alpha : H^q(X, A) \otimes H^r(X, B) \longrightarrow H^{q+r}(X, A \cup B),$$

where d^* is the cohomology homomorphism induced by

$$d : (X, A \cup B) \longrightarrow (X^2, X \times B \cup A \times X),$$

the obvious diagonal mapping. The cross and cup products are natural, bilinear, and associative. Since they are associative, they may be extended uniquely to n variables; then

$$d_n^*(u_1 \times \ldots \times u_n) = u_1 \cup \ldots \cup u_n.$$

The concept of a functional cohomology operation is fundamental throughout this paper. The basic reference on the subject is [14].

Let $\mathscr{A}_p$ be the Steenrod algebra [11] of stable primary cohomology operations of one variable using cohomology with coefficients $\mathbf{Z}_p$. Let $a \in \mathscr{A}_p$ with $\deg a = k$, so that $a : H^q(X, A; \mathbf{Z}_p) \longrightarrow H^{q+k}(X, A; \mathbf{Z}_p)$. For any element $w \in H^r(X^n, T^nX; \mathbf{Z}_p)$ with $d^*w = 0$ and $aw = 0$ we may define the functional cohomology operation $a_d w \in H^{r+k-1}(X, *; \mathbf{Z}_p)/\operatorname{Im} a + \operatorname{Im} d^*$. In particular, let $u_i \in H^{q_i}(X, *; \mathbf{Z}_p)$, $i = 1, \ldots, n$, with $\sum_i q_i = r$. If

$$u_1 \cup \ldots \cup u_n = d^*(u_1 \times \ldots \times u_n) = 0$$

and $a(u_1 \times \ldots \times u_n) = 0$, then $a_d(u_1 \times \ldots \times u_n)$is defined. The explicit definition uses the following diagram.

$$\begin{tikzcd}
& H^{r-1}(X, *) \arrow[r, "\delta"] \arrow[d, "a"] & H^r(X^n, X_d \cup T^nX) \arrow[r, "j^*"] \arrow[d, "a"] & H^r(X^n, T^nX) \arrow[r, "d^*"] \arrow[d, "a"] & H^r(X, *) \\
H^{r+k-1}(X^n, T^nX) \arrow[r, "d^*"] & H^{r+k-1}(X, *) \arrow[r, "\delta"] & H^{r+k}(X^n, X_d \cup T^nX) \arrow[r, "j^*"] & H^{r+k}(X^n, T^nX) &
\end{tikzcd}$$

Here X_d is the subspace $d(X)$ of X^n, and the rows are parts of the exact cohomology sequence of the triad $(X^n; X_d, T^nX)$.

DEFINITION (1.1). *If* $u_1 \cup \dots \cup u_n = 0$ *and* $a(u_1 \times \dots \times u_n) = 0$, *then we define the diagonal secondary cohomology operation* a_d *as follows*:

$$a_d(u_1 \times \dots \times u_n) = \delta^{-1} a\, j^{*-1}(u_1 \times \dots \times u_n)$$
$$\in H^{r+k-1}(X, *; \mathbf{Z}_p)/(aH^{r-1}(X, *; \mathbf{Z}_p) + d^*H^{r+k-1}(X^n, T^nX; \mathbf{Z}_p)).$$

It is easy to show that $a_d(u_1 \times \dots \times u_n)$ is a coset of $\operatorname{Im} a + \operatorname{Im} d^*$ in $H^{r+k-1}(X, *; \mathbf{Z}_p)$.

$a_d(u_1 \times \dots \times u_n)$ may be interpreted as a secondary cohomology operation in the variables $u_1, \dots, u_n$. As such, a_d is related to the primary relation given by the *Cartan formula* of a. For example, consider the Steenrod reduced power operation $P^k \in \mathscr{A}_p$, where $P^k = \mathscr{P}_p^k$ if p is odd or $P^k = Sq^k$ if $p = 2$. P^k satisfies the Cartan formula ([5] and [16] 6.12)

$$P^k(u \times v) = \textstyle\sum_{j=0}^k P^j u \times P^{k-j} v. \tag{1.2}$$

An alternate form is

$$P^k d^*(u \times v) - d^*(\textstyle\sum_{j=0}^k P^j \times P^{k-j})(u \times v) = 0,$$

where $d^*(u \times v) = u \cup v$ as usual, and from this we may get a relation on primary operations,

$$(P^k, d^*) \cdot (d^*, -\textstyle\sum_{j=0}^k P^j \times P^{k-j}) = 0. \tag{1.3}$$

Associated with this relation, we expect to find a secondary operation defined on $\operatorname{Ker} d^* \cap \operatorname{Ker} \sum_{j=0}^k P^j \times P^{k-j}$, with indeterminacy $\operatorname{Im} P^k + \operatorname{Im} d^*$, and in fact, $P_d^k(u \times v)$ is such an operation.

The generalization of a_d to relative cohomology is straightforward. For example, if $d: (X, A \cup B) \longrightarrow (X \times Y, X \times B \cup A \times Y)$ is used, for any $u \in H^q(X, A)$ and $v \in H^r(X, B)$ such that $u \cup v = 0$ and $a(u \times v) = 0$, $a_d(u \times v) \in H^{q+r+k-1}(X, A \cup B)/\operatorname{Im} a + \operatorname{Im} d^*$.

PROPOSITION (1.4). a_d *is natural. That is, if* $v_i \in H^{q_i}(Y,*)$, $i = 1, \dots, n$, *such that* $v_1 \cup \dots \cup v_n = 0$ *and* $a(v_1 \times \dots \times v_n) = 0$, *and* $f: X \longrightarrow Y$, *then*

$$f^* a_d(v_1 \times \dots \times v_n) = a_d(f^*v_1 \times \dots \times f^*v_n),$$

modulo the indeterminacy $\operatorname{Im} a + \operatorname{Im} d^*$.

As usual, the statement that "$x = y$ modulo I", where I is a subgroup of a group G, and x and y are elements or sets of elements of G, means that x and y are contained in the same coset of I, called the *indeterminacy*. With this convention, (1.4) follows from the naturality of the cross product and the naturality of the functional operation a_d.

For the next proposition, we recall the definitions of the Liusternik–Shnirel'man *category* and the *weak category* of a space. Let $p: (X^n, T^nX) \longrightarrow (X^{(n)}, *) = (X^n/T^nX, *)$ be the identification mapping which collapses T^nX to a point.

DEFINITION (1.5). cat $X < n$ *if and only if the diagonal mapping* d *of* X *is homotopic to a mapping* $d' : (X, *) \longrightarrow (X^n, T^n X)$ *with* $d'(X) \subseteq T^n X$. w cat $X < n$ *if and only if* $pd : (X, *) \longrightarrow (X^{(n)}, *)$ *is homotopic to the constant mapping.*

Clearly, w cat $X \leqq$ cat X. This definition of category and weak category reduces by one the values given by the classical definition ([2], Definitions (2.1) and (2.2)).

PROPOSITION (1.6). *If* $u_i \in H^{q_i}(X, *; \mathbf{Z}_p)$ *such that* $a_d(u_1 \times \ldots \times u_n) \neq 0$ *modulo the indeterminacy* $\operatorname{Im} a + \operatorname{Im} d^*$, *then* cat $X \geqq$ w cat $X \geqq n$.

Proof. Since $(X^n, T^n X)$ has the same homotopy type as a CW-complex pair, $p : (X^n, T^n X) \longrightarrow (X^{(n)}, *)$ induces isomorphisms on cohomology. Let $w = p^{*-1}(u_1 \times \ldots \times u_n) \in H^r(X^{(n)}, *)$. Then if pd is homotopic to the constant mapping, we have

$$a_d(u_1 \times \ldots \times u_n) = a_d(p^* w) = a_{pd}\, w = 0,$$

modulo $\operatorname{Im} a + \operatorname{Im} d^*$. Here we have used naturality of functional operations relative to the commutative diagram

$$\begin{array}{ccc} (X, *) & \xrightarrow{1} & (X, *) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle pd} \\ (X^n, T^n X) & \xrightarrow{p} & (X^{(n)}, *). \end{array}$$

Proposition (1.6) is a generalization to diagonal secondary operations of the theorem that, in a space of category less than n, any normal primary operation of n variables is identically zero ([15], Remarks pp. 179–180).

§2. THE PETERSON-STEIN FORMULAS

Theorems (2.3) and (2.4) give two formulas very useful for computation with diagonal operations. These formulas are the analogues for diagonal operations of the formulas (6.1) and (6.3) of Peterson and Stein [12].

Every cohomology operation $a \in \mathscr{A}_p$ satisfies a Cartan formula

(2.1) $$a(u \times v) = \textstyle\sum_{j \in I} b^j u \times c^j v,$$

or equivalently (by applying d^* and using naturality of a)

$$a(u \cup v) = \textstyle\sum_{j \in I} b^j u \cup c^j v,$$

which may be reformulated as a "scalar product"

(2.2) $$(a, d*) \cdot (d*, -\textstyle\sum_{j \in I} b^j \times c^j) = 0.$$

Here I is a finite index set, $b^j \in \mathscr{A}_p$, $c^j \in \mathscr{A}_p$, $\deg a = k$, $\deg b^j = k_j$, and $\deg c^j = k - k_j$.

Let $f : X \longrightarrow Y$ and take $u \in H^q(Y, *; \mathbf{Z}_p)$, $v \in H^r(Y, *; \mathbf{Z}_p)$.

THEOREM (2.3). *Assume*

(i) *for every* $j \in I$, $b^j u = 0$ *or* $c^j v = 0$,

(ii) $u \cup v = 0$, *and*

(iii) $f^* u = 0$. *Then*

$$f^* a_d (u \times v) = -a(u \cup_f v) + \textstyle\sum_{j \epsilon J} (b^j{}_\epsilon u) \cup f^* c^j v,$$

modulo the indeterminacy $\operatorname{Im} f^*d^* + \operatorname{Im} f^*a + \operatorname{Im} ad^* + \sum_{j \in J} \operatorname{Im} b^j \cup f^*c^j v$, *where* J *is the set of indices* $j \in I$ *for which* $c^j v \neq 0$.

THEOREM (2.4). *Assume*

(i) $f^*u \cup f^*v = 0$,

(ii) *for every* $j \in I$, $b^j f^*u = 0$ *or* $c^j f^*v = 0$, *and*

(iii) *for every* $j \in I$, $b^j u \cup c^j v = 0$. *Then*

$$a_d(f^*u \times f^*v) = a_f(u \cup v) - \textstyle\sum_{j \in I} b^j u \cup_f c^j v,$$

modulo the indeterminacy $\operatorname{Im} a + \operatorname{Im} d^* + \operatorname{Im} f^*$.

The proofs of (2.3) and (2.4) will be given in §6.

Note that the two formulas are approximately dual to each other. The indeterminacy of each is related to the domain of the other. Apparently this duality can be made precise and extended to the proofs, by using the concept of duality in an abelian category and a slightly different formulation of the theorems.

In order to treat some examples, consider a composition of mappings

$$K \xrightarrow{f} L \xrightarrow{g} X. \tag{2.5}$$

By using mapping cylinders, if necessary, we may replace L and X by spaces of the same based homotopy type such that f and g become inclusions. We then have the commutative diagram

$$\begin{array}{ccccc} (L, K) & \xrightarrow{G'} & (X, K) & \xrightarrow{F'} & (X, L) \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h''} \\ (L \cup_f cK, *) & \xrightarrow{G} & (X \cup_{gf} cK, *) & \xrightarrow{F} & (X \cup_g cL, *), \end{array} \tag{2.6}$$

where G' and F' are inclusion mappings, h', h, and h'' are identification mappings, and the second row is obtained from the first (up to homotopy type) by collapsing the subspaces. $L \cup_f cK$ is the space obtained by attaching the cone cK over K to L along the base K by identifying each point of K with its image under f. In many cases we may apply (2.3) to G, or (2.4) to F, to compute a_d in $H^*(X \cup_{gf} cK, *)$.

EXAMPLE (2.7). Let $X = S^q \vee S^{q'}$, $L = S^{r-1}$ where $r = q + q'$, and $K = S^{m-1}$. Let $g = [i_1, i_2] : S^{r-1} \longrightarrow S^q \vee S^{q'}$ be the Whitehead product of the inclusions of S^q and $S^{q'}$ in $S^q \vee S^{q'}$. Suppose there exists $a \in \mathscr{A}_p$ of positive degree such that $a : H^{r-1}(S^{r-1} \cup_f E^m, *) \longrightarrow H^m(S^{r-1} \cup_f E^m, *)$ is non-zero, or equivalently $a_f \neq 0$. (By [1] Theorem (1.1.1), $a = Sq^1, Sq^2, Sq^4$ or Sq^8 if $p = 2$, and by [9] or [10] Theorem (1.2.1), $a = \mathscr{P}^1_p$ if p is odd.) We have

$$S^{r-1} \cup_f E^m \xrightarrow{G} S^q \vee S^{q'} \cup_{gf} E^m \xrightarrow{F} S^q \vee S^{q'} \cup_g E^r.$$

Let v and v' be non-zero cohomology classes in $H^*(S^q \vee S^{q'} \cup_g E^r, * \,; Z_p)$ carried by S^q and $S^{q'}$ respectively, and let $u = F^*v$, $u' = F^*v'$. Since $g = [i_1, i_2]$, $S^q \vee S^{q'} \cup_g E^r = S^q \times S^{q'}$ so $v \cup v' \neq 0$. Consequently $u \cup_G u' \neq 0$, so $a(u \cup_G u') \neq 0$, and by (2.3)

$$\begin{aligned} G^* a_d(u \times u') &= -a(u \cup_G u') + \textstyle\sum_{j \in J} (b^j{}_G\, u) \cup G^* c^j u' \\ &= -a(u \cup_G u') \neq 0, \end{aligned}$$

with zero indeterminacy. On the other hand, $a_F(v \cup v') \neq 0$, so by (2.4),

$$a_d(u \times u') = a_d(F^*v \times F^*v') = a_F(v \cup v') - \textstyle\sum_{j \in I} b^j v \cup_F c^j v'$$
$$= a_F(v \cup v') \neq 0,$$

again with zero indeterminacy. Since $\operatorname{cat} S^q \vee S^{q'} = 1$, Proposition (1.6) implies that $\operatorname{cat} S^q \vee S^{q'} \cup_{gf} E^m = \operatorname{w\,cat} S^q \vee S^{q'} \cup_{gf} E^m = 2$.

EXAMPLE (2.8). Let $f: S^{m-1} \longrightarrow S^{r-1}$ be as in the preceding example, $r = 2q > 2$, and let $g : S^{r-1} \longrightarrow S^q$ be a mapping with (classical) Hopf invariant prime to p. (Then $q = 1, 2, 4$, or 8 if $p = 2$, and q is even if p is odd.) We have

$$S^{r-1} \cup_f E^m \xrightarrow{G} S^q \cup_{gf} E^m \xrightarrow{F} S^q \cup_g E^r.$$

Take a generator $v \in H^q(S^q \cup_g E^r, *; \mathbf{Z}_p)$, and let $u = F^*v$. Then $v \cup v \neq 0$, and $u \cup_G u \neq 0$. As in the preceding example, (2.3) yields

$$G^*a_d(u \times u) = -a(u \cup_G u) + 0 \neq 0.$$

(The sum of cup products vanishes since all cup products in $H^*(S^{r-1} \cup_f E^m, *)$ are zero, since $r - 1$ is odd.) Applying (2.4) to this example is more difficult.

In §4, a space $S^3 \cup E^7$ with $u \in H^3(S^3 \cup E^7, *; \mathbf{Z}_2)$ such that $Sq_d^2(u \times u) \neq 0$ will be constructed. This example cannot be treated by the above methods.

In the above examples, $g \not\simeq 0$ holds since there is a non-zero functional cup product induced by g, and $f \not\simeq 0$ since $a_f \neq 0$. Although $f \not\simeq 0$ and $g \not\simeq 0$ do not imply $gf \not\simeq 0$, in these cases we may conclude that $gf \not\simeq 0$ because of a non-vanishing *functionalized* diagonal operation induced by gf.

DEFINITION (2.9). *Let $f: X \longrightarrow Y$, $u \in H^q(Y, *)$, and $v \in H^{q'}(Y, *)$. Assume f is an inclusion, and let*

$$\cdots \longrightarrow H^{q-1}(Y, *) \xrightarrow{f^*} H^{q-1}(X, *) \xrightarrow{\delta} H^q(Y, X) \xrightarrow{j^*} H^q(Y, *) \longrightarrow \cdots$$

be the exact cohomology sequence of the pair (Y, X). The functionalized diagonal operation $(a_d)_f(u \times v)$ is defined

$$(a_d)_f(u \times v) = \delta^{-1} a_d(j^{*-1}u \times j^{*-1}v) \subseteq H^{q+q'+k-2}(X, *).$$

Of course, $(a_d)_f(u \times v)$ may be empty.

PROPOSITION (2.10). *$(a_d)_f(u \times v)$ is natural. That is, if $f' : X' \longrightarrow Y'$ is another mapping, and there are mappings g and g' such that the diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g"] & Y' \arrow[d, "g'"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes, then $g^*(a_d)_f(u \times v) \subseteq (a_d)_{f'}(g'^*u \times g'^*v)$. Hence if $(a_d)_f(u \times v)$ is non-empty and does not contain 0, $f \not\simeq 0$.

The proof is similar to the standard proof for primary cohomology operations.

Now (2.10) and the computations of diagonal operations in Examples (2.7) and (2.8) show that in these cases $gf \not\simeq 0$. For Example (2.7), this conclusion is a special case of a theorem of Hilton ([6], Theorem (A), p. 155) which states that $g_* : \pi_*(S^{r-1}) \to \pi_*(S^q \vee S^{q'})$ is a monomorphism onto a direct summand. For Example (2.8), the conclusion that $gf \not\simeq 0$ is classical when g is a fibre mapping, and in general it is included in a result of Serre ([13] Proposition (5), p. 281).

EXAMPLE (2.11). Let $0 \neq a \in \mathscr{A}_p$ with $\deg a > 1$. We shall construct a space with $a_d \neq 0$. Take integers q and q', both greater than 1, such that $r = q + q' > \deg a$. Let M be the mapping cylinder of $f: S^{r-1} \longrightarrow K(\mathbf{Z}_p, r-1)$, where $[f]$ generates $\pi_{r-1}(K(\mathbf{Z}_p, r-1),*)$. Let $g = [i_1, i_2] : S^{r-1} \longrightarrow S^q \vee S^{q'}$ as before, and let $S^q \vee S^{q'} \cup_g M$ be the space obtained by attaching M along S^{r-1} by g. Then there are natural mappings

$$M \xrightarrow{G} S^q \vee S^{q'} \cup_g M \xrightarrow{F} S^q \vee S^{q'} \cup_g E^r.$$

Under appropriate choice of $u \in H^q(S^q \vee S^{q'} \cup_g M, *)$ and $u' \in H^{q'}(S^q \vee S^q \cup_g M, *)$, $u \cup_G u'$ will be the fundamental class of M, and since $\deg a < r$, $a(u \cup_G u') \neq 0$. Now by applying (2.3) and dimension and naturality arguments, we get

$$G^* a_d(u \times u') = a(u \cup_G u') \neq 0$$

with zero indeterminacy.

The theorems and examples of this section could be modified to treat diagonal operations of more than two variables. However, by increasing the indeterminacy to include $\operatorname{Im} d_2{}^*$, we may express $a_{d_n}(u_1 \times \dots \times u_n)$ as a binary diagonal operation, by the following proposition.

PROPOSITION (2.12). *Let $u_i \in H^*(X,*)$ with $u_1 \cup \dots \cup u_n = 0$ and $a(u_1 \times \dots \times u_n) = 0$, where $a \in \mathscr{A}_p$. Then if $1 \leq j < n$, $a_{d_n}(u_1 \times \dots \times u_n) = a_{d_2}((u_1 \cup \dots \cup u_j) \times (u_{j+1} \cup \dots \cup u_n))$, modulo* $\operatorname{Im} a + \operatorname{Im} d_2^*$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
(X,*) \arrow[r, "1"] \arrow[d, "d_2"] & (X,*) \arrow[d, "d_n"] \\
(X,*) \times (X,*) \arrow[r, "d_j \times d_{n-j}"] & (X,*)^n.
\end{tikzcd}$$

By naturality we obtain

$$\begin{aligned} a_{d_n}(u_1 \times \dots \times u_n) &= a_{d_2}(d_j^*(u_1 \times \dots \times u_j) \times d_{n-j}{}^*(u_{j+1} \times \dots \times u_n)) \\ &= a_{d_2}((u_1 \cup \dots \cup u_j) \times (u_{j+1} \cup \dots \cup u_n)), \end{aligned}$$

modulo $\operatorname{Im} a + \operatorname{Im} d_2^*$. Q.E.D.

A similar result obviously applies to any partition of the set of indices.

Proposition (2.12) can be used to generalize Examples (2.7) and (2.11) to n variables. Replace $S^q \vee S^{q'}$ by a product of n spheres minus the interior of one top-dimensional cell in a CW-decomposition, and let g be the attaching map of the missing cell.

§3. TERNARY DIAGONAL OPERATIONS

We have just seen, in (2.12), that an n-ary diagonal operation $a_{d_n}(u_1 \times \ldots \times u_n)$ may be reduced to a binary diagonal operation $a_{d_2}((u_1 \cup \ldots \cup u_j) \times (u_{j+1} \cup \ldots \cup u_n))$, at the expense of increasing the indeterminacy to include Im d_2^*. Theorem (3.1) gives another way of reducing a ternary diagonal operation a_{d_3} to an expression involving binary diagonal operations.

Let $a \in \mathscr{A}_p$. Then a satisfies a Cartan formula (2.1).

THEOREM (3.1). *Let $u_i \in H^{q_i}(X, *; \mathbf{Z}_p)$, $i = 1, 2, 3$. Let J be a subset of I. Assume*

(i) $u_1 \cup u_2 = 0$;

(ii) $b^j(u_1 \times u_2) = 0$ for $j \in J$;

(iii) $c^j u_3 = 0$ for $j \in I - J$.

Then $a_{d_3}(u_1 \times u_2 \times u_3) = \sum_{j \in J} b^j_{d_2}(u_1 \times u_2) \cup c^j u_3$,

modulo Im a + Im $d_3^* + \sum_{j \in J} b^j H^{q_1+q_2-1}(X, *; \mathbf{Z}_p) \cup c^j u_3$.

The proof will be given at the end of §6.

We have stated (3.1) for the three variable case for greater clarity. A similar result holds more generally. Let $u_i \in H^{q_i}(X, *; \mathbf{Z}_p)$, $i = 1, \ldots, n$, $\sum_i q_i = r$, and let J be a subset of I.

THEOREM (3.2). *Assume*

(i) $u_1 \cup \ldots \cup u_s = 0$, *for some* s, $1 < s < n$

(ii) $b^j(u_1 \times \ldots \times u_s) = 0$ for $j \in J$

(iii) $c^j(u_{s+1} \times \ldots \times u_n) = 0$ *for* $j \in I - J$.

Then

$a_{d_n}(u_1 \times \ldots \times u_n) = \sum_{j \in J} (b^j)d_s(u_1 \times \ldots \times u_s) \cup c^j(u_{s+1} \cup \ldots \cup u_n)$,

modulo Im a + Im $d_n^* + \Sigma_{j \in J}$ (Im b^j) $\cup\, c^j(u_{s+1} \cup \ldots \cup u_n)$.

The proof of (3.2) is a straightforward generalization of that of (3.1), and is omitted.

As an application of (3.1), we consider sphere bundles over spheres with non-vanishing diagonal operations. Let B be a fibre bundle with base S^m and fibre S^q, $q > 1$. James and Whitehead ([8], 3.3) have given a decomposition of B as a CW-complex,

$$B = E^0 \cup E^q \cup E^m \cup E^{m+q} = S^q \cup E^m \cup E^{m+q}$$

where S^q is the fibre. Let $u \in H^q(B, *; \mathbf{Z}_p)$, such that $u \neq 0$ but $u \cup u = 0$. Let $P^k \in \mathscr{A}_p$ be a reduced power operation of degree $m + 1 - 2q > 0$. P^k satisfies the Cartan formula (1.2).

COROLLARY (3.3). *If $P^k_d(u \times u) \neq 0$, then $P^k_{d_3}(u \times u \times u) \neq 0$. Both have vanishing indeterminacy.*

Proof. Note that $P^j u = 0$ for $0 < j \leqq k$, since $H^s(B, *; \mathbf{Z}_p) = 0$ for $q < s < m$, and

$u \cup u = 0$ by assumption. Hence $P_d^k(u \times u)$ is defined. The indeterminacy is zero, for the same reason. Now by Theorem (3.1)

$$P_{d_3}^k(u \times u \times u) = P_d^k(u \times u) \cup P^0 u = P_d^k(u \times u) \cup u,$$

which is non-zero by Poincaré duality in the manifold B. Im $d_3^* = 0$ since all triple cup products vanish. It is obvious that $P^k : H^{3q-1}(B, *; Z_p) \longrightarrow H^{m+q}(B, *; Z_p)$ vanishes unless $m = 3q - 1$. In that case, let $e : B \longrightarrow S^m$ be the projection mapping, and let x be a generator of $H^m(S^m, *; Z_p)$. Then e^*x generates $H^m(B, *; Z_p)$ and $P^k e^* x = e^* P^k x = 0$.

COROLLARY (3.4). *If* $P_d^k(u \times u) \neq 0$, *then* cat $B =$ w cat $B = 3$.

Proof. Since a point has category 0, and attaching a cell can increase the category of a space by at most one, cat $B \leqq 3$. However, by (3.3), $P_{d_3}^k(u \times u \times u) \neq 0$, so by (1.6), cat $B \geqq$ w cat $B \geqq 3$. Hence cat $B =$ w cat $B = 3$.

§4. GENERALIZED HOPF INVARIANTS

Now we investigate the relationship of diagonal operations to generalized Hopf invariants. Since both of these invariants are used to study category, the existence of such a relationship is quite plausible. First, we review several generalized Hopf invariants.

G. W. Whitehead [17] gave the first generalization of the Hopf invariant. He defined a homomorphism $H : \pi_n(S^r) \longrightarrow \pi_n(S^{2r-1})$ for each $n < 3r - 3$. Blakers and Massey ([3], p. 202) extended the definition of H to cover the case $n = 3r - 3$. Hilton extended the definition to $n \leqq 4r - 4$ [7] and then to all values of n [6]. In [7, p. 413], Hilton also defined a generalized Hopf invariant H^*. Finally, Berstein and Hilton [2] generalized H to a "Hopf ϕ-invariant", and H^* to a "crude Hopf ϕ-invariant" $\bar{H}$.

We shall establish a relationship between H^* and diagonal operations, in Theorem (4.2). Although we state and prove the theorem for H^*, the statement and the proof generalize to $\bar{H}$ by making the obvious changes.

Hilton has noted in Theorem (6.2) of [6] that H^* is the suspension of H. We shall use this fact, and some computations of G. W. Whitehead, in giving several examples.

Next, we define H^*. Let $\phi : S^q \longrightarrow S^q \vee S^q$ be the identification mapping obtained by shrinking the equator of S^q to a point, so that the composition $S^q \xrightarrow{\phi} S^q \vee S^q \xrightarrow{i} S^q \times S^q$ is homotopic to the diagonal mapping, where i is the inclusion mapping. Now we have the exact homotopy sequence of the pair $(S^q \times S^q, S^q \vee S^q)$

$$\cdots \longrightarrow \pi_m(S^q \times S^q, S^q \vee S^q) \xrightarrow{\partial} \pi_{m-1}(S^q \vee S^q) \xrightarrow{i_*} \pi_{m-1}(S^q \times S^q) \longrightarrow \cdots$$

which splits [17, Theorem (4.8)], as we now show. Let $p_s : S^q \times S^q \longrightarrow S^q$ and $i_s : S_q \longrightarrow S^q \vee S^q$ be the projections and injections relative to the sth factor, $s = 1, 2$. Let $\lambda = i_{1*}p_{1*} + i_{2*}p_{2*}$. Then $i_*\lambda$ is the identity. Now λ and $Q = \partial^{-1}(1 - \lambda i_*)$ give the splitting of the sequence.

DEFINITION (4.1). [7]. *Let* $f: S^{m-1} \longrightarrow S^q$, *so that* $[f] \in \pi_{m-1}(S^q)$. *The generalized Hopf invariant of* f *is*

$$H^*f = h_* Q\varphi_*[f] \in \pi_m(S^{2q}),$$

where $h: (S^q \times S^q, S^q \vee S^q) \longrightarrow (S^q \times S^q/S^q \vee S^q, *) = (S^{2q}, *)$ *is the collapsing mapping.*

We also use the notation H^*f to denote any representative mapping of $H^*f \in \pi_m(S^{2q})$.

Now let $Y = S^q \cup_f E^m$, where $f: S^{m-1} \longrightarrow S^q$. Let $g: Y \longrightarrow S^m$ and $h': (Y^2, T^2 Y) \longrightarrow (Y^{(2)}, *)$ be the mappings collapsing S^q and $T^2 Y = Y \vee Y$ to a point, and let $j: S^q \longrightarrow Y$ and $j^{(2)}: (S^q)^{(2)} = S^{2q} \longrightarrow Y^{(2)}$ be the inclusions. Let $u \in H^q(Y, *; \mathbf{Z}_p)$ such that $a(u \times u) = 0$, where $a \in \mathscr{A}_p$ has degree $m + 1 - 2q$. Assume $(H^*f)^* j^{(2)*} h'^{*-1}(u \times u) = 0$.

THEOREM (4.2). $a_d(u \times u) = g^* a_{H^*f}\, j^{(2)*} h'^{*-1}(u \times u)$, *modulo* $g^* \operatorname{Im}(H^*f)^*$.

Note that $g^* \operatorname{Im}(H^*f)^*$ vanishes except possibly when $m = 2q$. The indeterminacy also includes the term $\operatorname{Im} a$, which vanishes, but the corresponding term would not always vanish in the generalization to $\bar{H}$.

The proof will be given after some examples.

EXAMPLE (4.3). G. W. Whitehead [17, Section 8] has constructed elements $x_q \in \pi_{2q}(S^q)$ with generalized Hopf invariant $H(x_q) = \nu_{2q-1} \in \pi_{2q}(S^{2q-1})$, provided that $q = 7$ or q is divisible by four, where ν_{2q-1} is the class of the $(2q - 3)$-fold suspension of the Hopf map $S^3 \longrightarrow S^2$. Let $Y_q = S^q \cup_{f_q} E^{2q+1}$, where $f_q \in x_q$, and let $u \in H^q(Y_q, *; \mathbf{Z}_2)$, $u \neq 0$. Now $(S^q)^{(2)} = S^{2q}$, and it is clear that $w = j^{(2)*} h'^{*-1} (u \times u)$ is non-zero in $H^{2q}(S^{2q}, *; \mathbf{Z}_2)$. Hence $Sq^2_{h_{2q}} w \neq 0$ provided $h_{2q} \in \nu_{2q} = H^*f_q$. By Theorem (4.2) we have

$$Sq^2_d(u \times u) = g^* Sq^2_{h_{2q}} w \neq 0 \in H^{2q+1}(S^q \cup_{f_q} E^{2q+1}, *; \mathbf{Z}_2).$$

In the same place, Whitehead also constructed elements $y_{8k} \in \pi_{16k+2}(S^{8k})$ with Hopf invariant $H(y_{8k}) = \bar{\nu}'_{16k-1} \in \pi_{16k+2}(S^{16k-1})$, where $\bar{\nu}'_{16k-1}$ is the $(16k - 5)$-fold suspension of an element of $\pi_7(S^4)$ of numerical Hopf invariant ± 1. Hence $H^*(y_8) = \bar{\nu}'_{16k}$, and $Sq^4_{h'_{16k}} \neq 0$ provided $h'_{16k} \in \bar{\nu}'_{16k}$. Let $Y'_{8k} = S^{8k} \cup_{g_{8k}} E^{16k+3}$, where $g_{8k} \in y_{8k}$, and let $0 \neq u \in H^{8k}(Y'_{8k}, *; \mathbf{Z}_2)$. Then, by (4.2), $Sq^4_d(u \times u) \neq 0$.

It follows from (1.6) that cat $Y_q =$ cat $Y'_{8k} = 2$, since the category of a sphere with a cell attached is at most two.

EXAMPLE (4.4). Consider the symplectic group $Sp(2)$. It may be fibred $Sp(2) \longrightarrow S^7$ with fibre $Sp(1) = S^3$. Hence there is a CW-decomposition $Sp(2) = S^3 \cup_f E^7 \cup E^{10}$. Now $Sp(2)$ is a principal bundle, and the characteristic mapping $S^6 \longrightarrow S^3 = Sp(1)$ is the attaching mapping f of E^7. Borel and Serre [4, p. 442] have shown that the generalized Hopf invariant $H(f)$ of f is nonzero, and hence it must be $\nu_5 \in \pi_6(S^5)$. Then $H^*(f) = \nu_6$, so if $u \in H^3(S^3 \cup_f E^7, *; \mathbf{Z}_2)$ with $u \neq 0$, then by Theorem (4.2) $Sq^2_d(u \times u) \neq 0$.

We may apply Corollaries (3.3) and (3.4) to $Sp(2)$. Take $v \in H^3(Sp(2), *; \mathbf{Z}_2)$ such that $i^*v = u$, where $i: S^3 \cup_f E^7 \longrightarrow Sp(2)$ is the inclusion. Then $Sq^2_d(v \times v) \neq 0$ by naturality of Sq^2_d. Then by (3.3), $Sq^2_{d_3}(v \times v \times v) \neq 0$, and by (3.4), cat $Sp(2) = 3$.

The remainder of this section is devoted to the proof of Theorem (4.2). We retain the notation introduced for Definition (4.1) and for the statement of Theorem (4.2).

Since the diagonal mapping of S^q is homotopic to $i\phi$, by the homotopy extension theorem there exists a mapping $d' : (Y, S^q) \longrightarrow (Y^2, S^q \vee S^q)$ such that $d'|S^q = \phi$ and

$$d \simeq j'd'k' : (Y, *) \longrightarrow (Y^2, T^2Y), \tag{4.5}$$

where $j' : (Y^2, S^q \vee S^q) \longrightarrow (Y^2, T^2Y)$ and $k' : (Y, *) \longrightarrow (Y, S^q)$ are inclusions, and d is the diagonal mapping of Y. Let $h' : (Y^2, T^2Y) \longrightarrow (Y^{(2)}, *)$ be the collapsing mapping, and let $k : (E^m, S^{m-1}) \longrightarrow (Y, S^q)$ be the natural mapping of the cell E^m onto its image in Y.

LEMMA (4.6). $h'j'd'k \simeq j^{(2)}H^*f : (E^m, S^{m-1}) \longrightarrow (Y^{(2)}, *)$.

We postpone the proof of the Lemma for the moment.

Proof of Theorem (4.2). We shall apply naturality in the homotopy-commutative diagram

$$\begin{tikzcd}
(S^m, *) & (E^m, S^{m-1}) \arrow[l, "g'"'] \arrow[r, "k"] \arrow[d, "H^*f"] & (Y, S^q) \arrow[d, "h'j'd'"] & (Y, *) \arrow[l, "k'"'] \arrow[d, "d"] \\
(S^{2q}, *) \arrow[from=u, "H^*f"] & (S^{2q}, *) \arrow[l, "1"'] \arrow[r, "j^{(2)}"] & (Y^{(2)}, *) & (Y^2, T^2Y). \arrow[l, "h'"']
\end{tikzcd}$$

In this diagram, g' is the mapping which collapses S^{m-1}. Clearly the first square homotopy-commutes. The second and third squares homotopy-commute by Lemma (4.6) and (4.5), respectively. Note that $g^* = k'^* k^{*-1} g'^*$. Now, by naturality in the diagram,

$$\begin{aligned} a_d(u \times u) &= k'^* k^{*-1} g'^* a_{H^*f} j^{(2)*} h'^{*-1}(u \times u) \\ &= g^* a_{H^*f} j^{(2)*} h'^{*-1}(u \times u), \end{aligned}$$

modulo the total indeterminacy, which is $\operatorname{Im} a + g^* \operatorname{Im} (H^*f)^*$.

The proof of Lemma (4.6) consists in chasing around Fig. 1.

$$\begin{tikzcd}
 & & \pi_m(Y^{(2)}, *) & \pi_m(S^{2q}, *) \arrow[l, "j^{(2)}_*"'] \\
\pi_m(Y, S^q) \arrow[r, "d'_*"] \arrow[d, "\partial"] & \pi_m(Y^2, S^q \vee S^q) \arrow[r, "\gamma'"] \arrow[ur, "(h'j')_*"] \arrow[d, "\partial"] & \pi_m(Y^2, S^q \vee S^q) \arrow[u, "(h'j')_*"] \arrow[d, "\partial"] & \pi_m((S^q, *)^2) \arrow[l, "j^2_*"'] \arrow[u, "h_*"] \arrow[dl, "\partial"] \\
\pi_{m-1}(S^q) \arrow[r, "\phi_*"] & \pi_{m-1}(S^q \vee S^q) \arrow[r, "\gamma"] & \pi_{m-1}(S^q \vee S^q) &
\end{tikzcd}$$

FIG. 1.

Here j^2 is the inclusion, ∂ is the boundary operator, and γ and γ' are homomorphisms to be defined. Let

$$\gamma = 1 - \lambda i_* = 1 - i_{1*}p_{1*}i_* - i_{2*}p_{2*}i_*$$

where i_s and p_s are the inclusions and projections introduced before (4.1). Let $i'_s : (Y, S^q) \longrightarrow (Y^2, S^q \vee S^q)$ and $p'_s : (Y^2, S^q \vee S^q) \longrightarrow (Y, S^q)$ be the inclusions and projections relative to the sth coordinate, $s = 1, 2$. Thus $i'_s|S^q = i_s$ and $p'_s|S^q \vee S^q = p_s i$, and consequently $\partial i'_{s*} = i_{s*}\partial$ and $\partial p'_{s*} = p_{s*}i_*\partial$. Now define

$$\gamma' = 1 - i'_{1*}p'_{1*} - i'_{2*}p'_{2*}.$$

We need the following subsidiary lemmas:

LEMMA (4.7). *Figure* 1 *is a commutative diagram.*

Proof. $\phi_*\partial = \partial d'_*$ and $\partial j^2_* = \partial$ by naturality of ∂. $j^{(2)}h = h'j'j^2 : (S^q, *)^2 \longrightarrow (Y^{(2)}, *)$. In the middle square,

$$\gamma\partial = (1 - i_{1*}p_{1*}i_* - i_{2*}p_{2*}i_*)\partial$$
$$= \partial(1 - i'_{1*}p'_{1*} - i'_{2*}p'_{2*}) = \partial\gamma'.$$

Finally, we show that $(h'j')_*\gamma = (h'j')_*$. Since $h'j'$ collapses T^2Y, $(h'j')_* i_s{}'_* = 0$. But then

$$(h'j')_*\gamma' = (h'j')_*(1 - i'_{1*}p'_{1*} - i'_{2*}p'_{2*}) = (h'j')_*.$$

LEMMA (4.8). Im $\gamma' \subseteq \operatorname{Im} j^2_*$

Proof. The triple $(Y^2, (S^q)^2, S^q \vee S^q)$ induces an exact sequence

$$\cdots \longrightarrow \pi_m((S^q, *)^2) \xrightarrow{j^2_*} \pi_m(Y^2, S^q \vee S^q) \xrightarrow{j''_*} \pi_m(Y^2, (S^q)^2) \longrightarrow \cdots$$

and there are projections $p''_s : (Y^2, (S^q)^2) \longrightarrow (Y, S^q)$ such that $p''_s j'' = p'_s$. Now let $x \in \pi_m(Y^2, S^q \vee S^q)$. Then

$$p''_{s*}j''_*\gamma'x = p'_{s*}(1 - i'_{1*}p'_{1*} - i'_{2*}p'_{2*})x$$
$$= p'_{s*}x - p'_{s*}x = 0,$$

for $s = 1, 2$. Hence $j''_*\gamma'x = 0$, so $\gamma'x \in \operatorname{Im} j^2_*$.

Proof of Lemma (4.6). The homotopy class $[k]$ of k is an element of $\pi_m(Y, S^q)$, and it suffices to show that

$$(h'j')_*d'_*[k] = j^{(2)}{}_*(H^*f),$$

for then

$$[h'j'd'k] = (h'j')_*d'_*[k] = j^{(2)}{}_*(H^*f) = [j^{(2)}H^*f],$$

so that $h'j'd'k$ and $j^{(2)}H^*f$ lie in the same homotopy class.

By (4.8), there exists $w \in \pi_m((S^q, *)^2)$ such that $\gamma' d'_*[k] = j^2_*w$. Now $\partial\ [k] = [f] \in \pi_{m-1}(S^q)$, since f is the attaching map of E^m and k is the natural mapping of (E^m, S^{m-1}) onto its image in (Y, S^q). Also $Q = \partial^{-1}(1 - \lambda i_*) = \partial^{-1}\gamma$, so

$$H^*f = h^*Q\phi_*[f] = h_*\partial^{-1}\gamma\phi_*[f],$$

which equals h_*w, since

$$\partial w = \partial j^2_* w = \partial\gamma' d'_*[k] = \gamma\phi_*[f]$$

and $\partial : \pi_m((S^q, *)^2) \longrightarrow \pi_{m-1}(S^q \vee S^q)$ is injective. Finally,

$$(h'j')_*d'_*[k] = (h'j')_*j^2{}_*w = j^{(2)}{}_*h_*w = j^{(2)}{}_*H^*f.$$

This completes the proof of Lemma (4.6) and hence of Theorem (4.2).

§5. FURTHER PROPERTIES OF DIAGONAL OPERATIONS

This section studies elementary properties of diagonal operations concerning multilinearity, suspension, and composition with primary operations.

PROPOSITION (5.1). *Let* $a \in \mathcal{A}_p$.

Then (i) $a_d(x_1 \times \dots \times x_n)$ *is multilinear;*

(ii) $a_d(x_1 \times \dots \times x_n)$ *is commutative, with the usual sign convention; that is,*

$$a_d(u_1 \times \dots \times u_{i+1} \times u_i \times \dots \times u_n) = (-1)^{q_i q_{i+1}} a_d(u_1 \times \dots \times u_n),$$

modulo $\operatorname{Im} a + \operatorname{Im} d_n^*$, *where* $q_j = \deg u_j$, $j = i, i+1$.

Proof. Multilinearity follows from the multilinearity of the cross product, and linearity of the functional operation a_d.

We establish commutativity for the case $n = 2$, $i = 1$, since the proof in the general case is similar. Let X be a space with subspaces A_1, A_2, and $A = A_1 \cup A_2$. Let $u_i \in H^{q_i}(X, A_i)$, $i = 1, 2$. There is a commutative diagram

$$\begin{tikzcd}
(X, A) \arrow[r, "1"] \arrow[d, "d"'] & (X, A) \arrow[d, "d"] \\
(X, A_1) \times (X, A_2) \arrow[r, "f"] \arrow[d, "p_i"'] & (X, A_2) \times (X, A_1), \arrow[d, "p_i'"] \\
(X, A_i) \arrow[r, "1"] & (X, A_i)
\end{tikzcd}$$

where f permutes the two factors, and p_i and p_i' are the projections. Now

$$\begin{aligned}
f^*(u_2 \times u_1) &= (f^* p_2'^* u_2) \cup (f^* p_1'^* u_1) \\
&= (-1)^{q_1 q_2} p_1^* u_1 \cup p_2^* u_2 \\
&= (-1)^{q_1 q_2} u_1 \times u_2.
\end{aligned}$$

Commutativity of a_d now follows from the naturality and linearity of a_d.

The question of stability of cohomology operations under suspension is important. The next proposition shows that diagonal operations are far from stable.

PROPOSITION (5.2). *Let* a_d *be a diagonal operation. Then* $s(a_d)$, *the suspension of* a_d, *is zero.*

Proof. Let $s : H^{r+1}(sX, *) \longrightarrow H^r(X, *)$ be the suspension isomorphism. Then, by definition

$$s(a_d)(u_1 \times \dots \times u_n) = s a_d(s^{-1} u_1 \times \dots \times s^{-1} u_n).$$

But $a_d(s^{-1} u_1 \times \dots \times s^{-1} u_n) = 0$ by (1.6,) since $\operatorname{cat} sX \leqq 1$.

Although the suspension of a_d vanishes, we now prove a property analogous to stability in one variable, relating a_d with itself through the coboundary δ. Let (Y, X) be a pair of spaces with inclusion mapping $f : X \longrightarrow Y$. Let $d : (Y, X) \longrightarrow (Y^2, Y \times * \cup X \times Y)$ and $d' : (X, *) \longrightarrow (X^2, T^2 X)$ be diagonal mappings. Finally, let $u \in H^q(X, *)$ and $v \in H^r(Y, *)$ such that $u \cup f^* v = 0$ and $a(u \times v) = 0$.

PROPOSITION (5.3). $\delta a_{d'}(u \times f^* v) = -a_d(\delta u \times v)$, *modulo* $\operatorname{Im} a + \operatorname{Im} d^* + \delta \operatorname{Im} d'^*$.

Proof. Theorem (13.2) of [14] can be adapted (as in Theorem (15.11) of [14]) by replacing the cup product $u \longrightarrow u \cup v$ by the operation a to obtain the following lemma.

LEMMA (5.4). *Let* $f : (X', A') \longrightarrow (X, A)$ *and let* $g : A' \longrightarrow A$ *be the map defined by* f. *If* $x \in H^s(A)$ *satisfies* $ax = 0$ *and* $g^*x = 0$, *then*

$$\delta' a_g x = -a_f \delta x,$$

modulo $\operatorname{Im} a + \operatorname{Im} f^*$, *where* δ *and* δ' *are the cohomology coboundary operators of* (X, A) *and* (X', A').

In order to apply the lemma to the triad mapping $d'' : (Y; X, *) \longrightarrow (Y^2; X \times Y, T^2 Y)$ induced by the diagonal mapping d, let h be the identification mapping collapsing $T^2 Y = Y \vee Y$ to a point as in the following diagram.

$$\begin{tikzcd}
(Y; X, *) \arrow[r, "d''"] \arrow[d, "1"] & (Y^2; X \times Y, T^2Y) \arrow[d, "h"] \\
(Y; X, *) \arrow[r, "hd''"] & (Y^2/T^2Y; X \times Y/X \vee Y, *)
\end{tikzcd}$$

h^* maps the cohomology sequence of $(Y^2/T^2 Y; X \times Y/X \vee Y, *)$ isomorphically onto that of $(Y^2; X \times Y, T^2 Y)$ and hd'' may be considered a map of pairs $(Y, X) \longrightarrow (Y^2/T^2 Y, X \times Y/X \vee Y)$. Applying the lemma to $x = h^{*-1}(u \times v) \in H^{q+r}(X \times Y/X \vee Y, *)$, we get

$$\delta a_{h(1 \times f)d'} x = -a_{hd''} \delta'' x,$$

modulo $\operatorname{Im} a + \operatorname{Im} d''^* h^*$, where δ and δ'' are the coboundary operators of (Y, X) and $(Y^2/T^2 Y, X \times Y/X \vee Y)$. Since h^* is an isomorphism,

$$\delta a_{(1 \times f)d'}(u \times v) = -a_d \delta'(u \times v),$$

modulo the indeterminacy, where $\delta' = \delta \times 1$ is the coboundary operator of the triad $(Y^2; X \times Y, T^2 Y)$. Finally, by naturality,

$$\delta a_{d'}(u \times f^* v) = -a_d \delta'(u \times v) = -a_d(\delta u \times v).$$

It is straightforward to check the indeterminacy. This completes the proof.

Now let $a^j \in \mathscr{A}_p$ and $b^j \in \mathscr{A}_p$, and suppose $\deg a^j = k_j$ and $\deg b^j = k - k_j$, where j is in the finite index set J. $\sum_{j \in J} a^j b^j$ is an element of $\mathscr{A}_p$ of degree k. Let $u_i \in H^*(X, *)$ such that $u_1 \cup \dots \cup u_n = 0$ and $b^j(u_1 \times \dots \times u_n) = 0$ for each $j \in J$.

PROPOSITION (5.5). *If* $\sum_{j \in J} a^j b^j = 0$, *then*

$$\textstyle\sum_{j \in J} a^j b^j_d(u_1 \times \dots \times u_n) = 0,$$

modulo the indeterminacy $\sum_{j \in J} a^j \operatorname{Im} b^j + \operatorname{Im} d^*$.

The proof is straightforward. In particular, let $a, b \in \mathscr{A}_p$, so that $(1)(ab) - (a)(b) = 0$.

COROLLARY (5.6). $(ab)_d(u_1 \times \dots \times u_n) = a(b_d)(u_1 \times \dots \times u_n)$, *modulo* $\operatorname{Im} ab + \operatorname{Im} d^*$.

Let $a, a' \in \mathscr{A}_p$ and suppose that a satisfies the Cartan formula (2.1). Assume $u, v \in H^*(X, *)$ such that $u \cup v = 0$, $b^j u \cup c^j v = 0$, and $a'(b^j u \times c^j v) = 0$ for $j \in I$.

PROPOSITION (5.7). $(a'a)_d(u \times v) = \sum_{j \in I} a'_d(b^j u \times c^j v)$, *modulo* $\operatorname{Im} a' + \operatorname{Im} d*$.

Proof. It is an elementary property of functional operations that $(a'a)_d = a'_d a$ modulo $\operatorname{Im} a' + \operatorname{Im} d^*$. The proposition follows from the Cartan formula of a.

§6. PROOFS OF THE MAIN THEOREMS

First we prove a lemma about functional operations on product spaces. Let $f: X \longrightarrow Y$ and $f': X' \longrightarrow Y'$ be mappings, $u \in H^q(Y, *; \mathbf{Z}_p)$, $v \in H^{q'}(Y', *; \mathbf{Z}_p)$, and $a \in \mathscr{A}_p$ with Cartan formula (2.1).

LEMMA (6.1). *Let J be a subset of I. Assume*

(i) $f^*u = 0$

(ii) $b^i u = 0$ for $i \in J$

(iii) $c^i v = 0$ *for* $i \in J - I$. *Then*

$$a_{f \times f'}(u \times v) = \textstyle\sum_{i \in J}(b^i_f u) \times (f'^* c^i v),$$

modulo the total indeterminacy $\operatorname{Im} a + \operatorname{Im}(f \times f')^* + \sum_{i \in J}(\operatorname{Im} b^i) \times (f'^* c^i v)$.

Proof. By using mapping cylinders, we assume that f and f' are inclusions, so that (Y, X) and (Y', X') are pairs.

First, consider the case in which $X' = Y'$ and f' is the identity. $a_{f \times 1}$ is defined using the sequence

$$\ldots \longrightarrow H^*((X, *) \times (Y', *)) \xrightarrow{\delta'} H^*((Y, X) \times (Y', *)) \xrightarrow{(j \times 1)^*} H^*((Y, *) \times (Y', *)) \xrightarrow{(f \times 1)^*} \ldots.$$

Since $f^* u = 0$, there exists $u_0 \in H^q(Y, X; \mathbf{Z}_p)$ such that $j^* u_0 = u$. For $i \in J$, $b^i u = 0$, so there exists $x_i \in H^*(X, *; \mathbf{Z}_p)$ with $\delta x_i = b^i u_0$. Then x_i represents $b^i_f u$.

Now $(j \times 1)^*(u_0 \times v) = j^* u_0 \times v = u \times v$, and

$$\begin{aligned} a(u_0 \times v) &= \textstyle\sum_{i \in I} b^i u_0 \times c^i v \\ &= \textstyle\sum_{i \in J} \delta x_i \times c^i v \\ &= \delta' \textstyle\sum_{i \in J} x_i \times c^i v \end{aligned}$$

so that $\sum_{i \in J} x_i \times c^i v$ represents $a_{f \times 1}(u \times v)$. Hence

$$a_{f \times 1}(u \times v) = \textstyle\sum_{i \in J} b^i_f u \times c^i v.$$

The general case follows by naturality. For

$$\begin{aligned} a_{f \times f'}(u \times v) &= (1 \times f')^* a_{f \times 1}(u \times v) \\ &= \textstyle\sum_{i \in J}(b^i_f u) \times (f'^* c^i v). \end{aligned}$$

It is easy to check the indeterminacy.

Before proving (2.3) and (2.4), we need a formulation of the functional cup product different from the usual one. Let $f: X \longrightarrow Y$ be an inclusion mapping. Then we have the following diagram.

$$\begin{tikzcd} & H^{r-1}(X^2, T^2X) \arrow[r, "\delta"] \arrow[d, "d'^*"] & H^r(Y^2, X^2 \cup T^2Y) \arrow[r, "j'^*"] \arrow[d, "d^*"] & H^r(Y^2, T^2Y) \arrow[r, "f^{2*}"] \arrow[d, "d^*"] & H^r(X^2, T^2X) \\ H^{r-1}(Y, *) \arrow[r, "f^*"] & H^{r-1}(X, *) \arrow[r, "\delta"] & H^r(Y, X) \arrow[r, "j^*"] & H^r(Y, *) & \end{tikzcd} \tag{6.2}$$

Here d and d' are the diagonal mappings of Y and X. Let $u \in H^q(Y, *)$, $v \in H^{q'}(Y, *)$, $r = q + q'$, such that $f^{2*}(u \times v) = f^* u \times f^* v = 0$ and $u \cup v = d^*(u \times v) = 0$.

DEFINITION (6.3). *Under these conditions, the functional cup product $u \cup'_f v$ is defined*

$$u \cup'_f v = \delta^{-1} d^* j'^{*-1}(u \times v) \in H^{r-1}(X, *)/(\operatorname{Im} d'^* + \operatorname{Im} f^*).$$

Now we relate $\cup'_f$ to the usual functional cup product. Recall that whenever $f^* u = 0$ and $u \cup v = 0$, the (left) functional cup product $u \cup_f v$ is defined

$$u \cup_f v = \delta^{-1}(j^{*-1} u \cup v) \in H^{r-1}(X, *)/(H^{q-1}(X, *) \cup f^* v + \operatorname{Im} f^*).$$

LEMMA (6.4). *Suppose that $f^* u = 0$ so that the (left) functional cup product $u \cup_f v$ is defined. Then $u \cup_f v = u \cup'_f v$, modulo* $\operatorname{Im} d'^* + \operatorname{Im} f^*$.

Proof. Since $f^* u = 0$, there exists $u' \in H^q(Y, X)$ with $j^* u' = u$. Now $j^*(u' \cup v) = u \cup v = 0$, so there exists $w \in H^{r-1}(X, *)$ with $\delta w = u' \cup v$. By definition of $u \cup_f v$, $w \in u \cup_f v$.

Now let $j'' : (Y^2, X^2 \cup T^2 Y) \longrightarrow (Y, X) \times (Y, *)$ be the inclusion mapping, so that $j'' j' = j \times 1 : (Y, *)^2 \longrightarrow (Y, X) \times (Y, *)$. Then

$$j'^*(j''^*(u' \times v)) = j^* u' \times v = u \times v,$$

and

$$\delta w = u' \cup v = d^*(u' \times v) = d^* j''^*(u' \times v).$$

Thus $w \in u \cup'_f v$, and the proof is complete.

The proof of Theorem (2.3) consists of chasing around the commutative diagrams of Figs. 2 and 3, and the proof of Theorem (2.4) consists of chasing around Figs. 4 and 5.

$$\begin{tikzcd}
H^{m-1}(Y^2, T^2 Y) \arrow[r, "f^{2*}"] \arrow[d, "1"] & H^{m-1}(X^2, T^2 X) \arrow[r, "\delta"] \arrow[d, "d'^*"] & H^m(Y^2, X^2 \cup T^2 Y) \arrow[r, "j^*"] \arrow[d, "h^*"] & H^m(Y^2, T^2 Y) \arrow[r, "f^{2*}"] \arrow[d, "1"] & H^m(X^2, T^2 X) \arrow[d, "d'^*"] \\
H^{m-1}(Y^2, T^2 Y) \arrow[r, "(df)^*"] & H^{m-1}(X, *) \arrow[r, "\delta"] & H^m(Y^2, X_d \cup T^2 Y) \arrow[r, "j^*"] & H^m(Y^2, T^2 Y) \arrow[r, "(df)^*"] & H^m(X, *) \\
H^{m-1}(Y^2, T^2 Y) \arrow[r, "d^*"] \arrow[u, "1"] & H^{m-1}(Y, *) \arrow[r, "\delta"] \arrow[u, "f^*"] & H^m(Y^2, Y_d \cup T^2 Y) \arrow[r, "j^*"] \arrow[u, "h^*"] & H^m(Y^2, T^2 Y) \arrow[r, "d^*"] \arrow[u, "1"] & H^m(Y, *) \arrow[u, "f^*"]
\end{tikzcd}$$

FIG. 2.

$$\begin{tikzcd}
H^{m-1}(Y^2, T^2 Y) \arrow[r, "(df)^*"] \arrow[d, "d^*"] & H^{m-1}(X, *) \arrow[r, "\delta"] \arrow[d, "1"] & H^m(Y^2, X_d \cup T^2 Y) \arrow[r, "j^*"] \arrow[d, "d^*"] & H^m(Y^2, T^2 Y) \arrow[r, "(df)^*"] \arrow[d, "d^*"] & H^m(X, *) \arrow[d, "1"] \\
H^{m-1}(Y, *) \arrow[r, "f^*"] & H^{m-1}(X, *) \arrow[r, "\delta'"] & H^m(Y, X) \arrow[r, "j^*"] & H^m(Y, *) \arrow[r, "f^*"] & H^m(X, *)
\end{tikzcd}$$

FIG. 3.

Each row in Figs. 2, 3, 4, and 5 is a segment of the exact cohomology sequence of some triad. Note that the middle row of Fig. 2 and the top row of Fig. 3 coincide, as do those of Figs. 4 and 5. δ, δ', and δ'' represent the coboundary operators of the triads. d^* and d'^*

denote any cohomology homomorphisms induced by the diagonal mappings of Y and X, respectively.† Since we assume (by using the mapping cylinder of f for Y) that f is an inclusion, all the mappings inducing homomorphisms in Figs. 2, 3, 4, and 5 are inclusions. Hence the diagrams are commutative.

Let $s = q + r$ and let $t = q + r + k$, where $u \in H^q(Y, *)$, $v \in H^r(Y, *)$, and $k = \deg a$. We are interested in the cases $m = s$ and $m = t$.

Proof of Theorem (2.3). We are given $u \times v \in H^s(Y^2, T^2 Y)$ with $d^*(u \times v) = 0$ and $f^*u = 0$. Hence there exist elements $x \in H^s(Y^2, Y_d \cup T^2 Y)$ and $y \in H^s(Y^2, X^2 \cup T^2 Y)$ such that $j^*x = u \times v$ and $j^*y = u \times v$. Now $a(u \times v) = 0$ by hypothesis (i), so $j^*ax = j^*ay = 0$. Therefore there exist $x' \in H^{t-1}(Y, *)$ and $y' \in H^{t-1}(X^2, T^2 X)$ such that $\delta x' = ax$ and $\delta y' = ay$. Clearly

$$x' \in a_d(u \times v) \tag{6.5}$$

and

$$y' \in a_{f^2}(u \times v) = \textstyle\sum_{j \in J}(b_f^j u) \times (f^* c^j v), \tag{6.6}$$

by (6.1).

Since $j^*(h^*y - h^*x) = 0$, there exists $z \in H^{s-1}(X, *)$ with $\delta z = h^*y - h^*x$. Now $\delta' z = d^* \delta z = d^*h^*y - d^*h^*x = (hd)^*y$, since $hd : (Y, X) \longrightarrow (Y^2, Y_d \cup T^2 Y)$ induces $(hd)^* = 0$. Moreover, $i^*y = u \times v$, so by (6.3),

$$z \in u \cup_f' v. \tag{6.7}$$

$$\begin{aligned} \text{Finally, } \delta(f^*x' - d'^*y' + az) &= h^*\delta x' - h^*\delta y' + a\delta z \\ &= ah^*x - ah^*y + a\delta z \\ &= a(h^*x - h^*y + \delta z) = 0, \end{aligned}$$

so $f^*x' = d'^*y' - az$, modulo $\operatorname{Im} f^*d^*$. Using (6.5), (6.6), (6.7), and Lemma (6.4) we obtain

$$\begin{aligned} f^*a_d(u \times v) &= d'^* \textstyle\sum_{j \in J}(b_f^j u) \times (f^* c^j v) - a(u \cup_f' v) \\ &= \textstyle\sum_{j \in J}(b_f^j u) \cup (f^* c^j v) - a(u \cup_f v), \end{aligned}$$

modulo $\operatorname{Im} f^*d^* + \operatorname{Im} f^*a + \operatorname{Im} ad'^* + \sum_{j \in J} \operatorname{Im} b^j \cup (f^* c^j v)$.

$$\begin{tikzcd} H^m(X, *) & H^m(Y, *) \arrow[l, "f^*"'] & H^m(Y, X) \arrow[l, "j^*"'] & H^{m-1}(X, *) \arrow[l, "\delta'"'] & H^{m-1}(Y, * \arrow[l, "f^*"'] \\ H^m(X, *) \arrow[u, "1"] \arrow[d, "1"'] & H^m(Y^2, T^2 Y) \arrow[l, "(df)^*"'] \arrow[u, "d^*"] \arrow[d, "f^{2*}"'] & H^m(Y^2, X_d \cup T^2 Y) \arrow[l, "j^*"'] \arrow[u, "d^*"] \arrow[d, "f^{2*}"'] & H^{m-1}(X, *) \arrow[l, "\delta"'] \arrow[u, "1"] \arrow[d, "1"'] & H^{m-1}(Y^2, T^2 Y \arrow[l, "(df)^*"'] \arrow[u, "d^*"] \arrow[d, "f^{2*}"'] \\ H^m(X, *) & H^m(X^2, T^2 X) \arrow[l, "d'^*"'] & H^m(X^2, X_d \cup T^2 X) \arrow[l, "j^*"'] & H^{m-1}(X, *) \arrow[l, "\delta''"'] & H^{m-1}(X^2, T^2 X) \arrow[l, "d'^*"'] \end{tikzcd}$$

FIG. 4.

† Note, however, that in the statements and applications of Theorems (2.3) and (2.4) we have denoted all diagonal mappings by d.

$$\begin{tikzcd}
H^m(X, *) & H^m(Y^2, T^2Y) \arrow[l, "(df)^*"'] & H^m(Y^2, X_d \cup T^2Y) \arrow[l, "j^*"'] & H^{m-1}(X, *) \arrow[l, "\delta"'] & H^{m-1}(Y^2, T^2Y) \arrow[l, "(df)^*"'] \\
H^m(X^2, T^2X) \arrow[u, "d'^*"] & H^m(Y^2, T^2Y) \arrow[l, "f^{2*}"'] \arrow[u, "1"] & H^m(Y^2, X^2 \cup T^2Y) \arrow[l, "j^*"'] \arrow[u, "h^*"] & H^{m-1}(X^2, T^2X) \arrow[l, "\delta"'] \arrow[u, "d'^*"] & H^{m-1}(Y^2, T^2Y) \arrow[l, "f^{2*}"'] \arrow[u, "1"]
\end{tikzcd}$$

FIG 5.

Proof of Theorem (2.4). We are given $u \times v \in H^s(Y^2, T^2Y)$ with $(df)^*(u \times v) = 0$ by hypothesis (i). Hence there exists $w \in H^s(Y^2, X_d \cup T^2Y)$ such that $j^*w = u \times v$. Now $f^{2*}(b^ju \times c^jv) = 0$ by (ii), so there exist elements $w_j \in H^t(Y^2, X^2 \cup T^2Y)$ with $j^*w_j = b^ju \times c^jv$. Since $j^*d^*h^*w_j = b^ju \cup c^jv = 0$ by (iii), there exist elements $z_j \in H^{t-1}(X, *)$ such that $\delta' z_j = d^*h^*w_j$. Then

$$z_j \in b^ju \cup'_f c^jv. \tag{6.8}$$

Since $j^*(aw - \sum_{j\in I} h^*w_j) = a(u \times v) - \sum_{j\in I} b^ju \times c^jv = 0$, there exists $z \in H^{t-1}(X, *)$ with $\delta z = aw - \sum_{j\in I} h^*w_j$. Now $\delta'' z = f^{2*}\delta z = af^{2*}w - 0$, since $hf^2 : (X^2, X_d \cup T^2X) \longrightarrow (Y^2, X^2 \cup T^2Y)$ induces $f^{2*}h^* = 0$. Also $j^*f^{2*}w = f^*u \times f^*v$, so

$$z \in a_{d'}(f^*u \times f^*v). \tag{6.9}$$

Now $\delta'(z + \sum_j z_j) = d^*(aw - \sum_j h^*w_j) + \sum_j d^*h^*w_j = a\, d^*w$, and $j^*d^*w = u \cup v$, so

$$z + \textstyle\sum_{j\in I} z_j \in a_f(u \cup v). \tag{6.10}$$

Finally, by (6.8), (6.9), (6.10), and Lemma (6.4),

$$a_{d'}(f^*u \times f^*v) = a_f(u \cup v) - \textstyle\sum_{j\in I} b^ju \cup_f c^jv,$$

modulo the indeterminacy $\operatorname{Im} a + \operatorname{Im} d'^* + \operatorname{Im} f^*$.

Proof of Theorem (3.1). Note that $d_3 = (d_2 \times 1)\, d_2$. Thus

$$\begin{aligned}
a_{d_3}(u_1 \times u_2 \times u_3) &= d_2^*\, a_{d_2\times 1}((u_1 \times u_2) \times u_3), \text{ by naturality,} \\
&= d_2^* \textstyle\sum_{j\in J} b^j_{d_2}(u_1 \times u_2) \times c^ju_3, \quad \text{by (6.1),} \\
&= \textstyle\sum_{j\in J} b^j_{d_2}(u_1 \times u_2) \cup c^ju_3.
\end{aligned}$$

Check the indeterminacy.

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